Theory of networked minority games based on strategy pattern dynamics

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(Received 16 June 2004; published 9 November 2004)

We formulate a theory of agent-based models in which agents compete to be in a winning group. The agents may be part of a network or not, and the winning group may be a minority group or not. An important feature of the present formalism is its focus on the dynamical pattern of strategy rankings, and its careful treatment of the strategy ties which arise during the system's temporal evolution. We apply it to the minority game with connected populations. Expressions for the mean success rate among the agents and for the mean success rate for agents with k neighbors are derived. We also use the theory to estimate the value of connectivity p above which the binary-agent-resource system with high resource levels makes the transition into the high-connectivity state.

DOI: 10.1103/PhysRevE.70.056102 PACS number(s): 02.50.Le, 05.65.+b, 05.40.−a, 89.90.+n

I. INTRODUCTION

Agent-based models form an important part of research on complex adaptive systems [1]. For example, the self-organization of an evolving population consisting of agents competing for a limited resource has potential applications in areas such as economics, biology, engineering, and social sciences [1,2]. The bar-attendance problem proposed by Arthur [3,4] constitutes a typical setting for such a system in which a population of agents decide whether to go to a popular bar having limited seating capacity. The agents are informed of the attendance in past weeks, and hence the agents share common information, interact through their actions, and learn from past experience. The problem can be simplified by considering binary games, either in the form of the minority game (MG) [5,6] or in the more general form of a binary-agent-resource (BAR) game [7,8]. For modest resource levels in which there are more losers than winners, the minority game proposed by Challet and Zhang [5,9] represents a simple, yet nontrivial, model that captures many of the essential features of such a competing population.

The MG considers an odd number N of agents. At each time step, the agents independently decide between two options 0 and 1. The winners are those who choose the minority option. The agents learn from past experience by evaluating the performance of their strategies, where each strategy maps the available global information (i.e., the record of the most recent m winning options) to an action. One important quantity in the MG is the standard deviation σ of the number of agents making a particular choice. This quantity reflects the performance of the population as a whole in that a small σ implies on average more winners per turn, and hence a higher success rate per turn per agent. In the MG, σ exhibits a nonmonotonic dependence on the memory size m of the agents [10–12]. When m is small, there is significant overlap between the agents' strategies. This crowd effect [7,13,14] leads to a large σ, implying the number of losers is high. This is the crowded, or informationally efficient, phase of the MG. In the informationally inefficient phase where m is large, σ is moderately small and the agents perform better than if they were to decide their actions randomly.

Theoretical analysis of the MG has been the focus of many studies [4,7,11–20]. Mapping the MG into the language of disordered spin systems makes the machinery in statistical physics of disordered system, most noticeably the replica trick, useful in the study of models of a competing population. For the MG, calculations based on the replica method work well for the case of a large strategy pool, i.e., when the strategy pool is much larger than the strategies actually being used in making decisions. This is referred to as the informationally inefficient phase because information is left in the resulting bit-string patterns for a single realization of the game. In the informationally efficient phase, the whole pool of strategies tends to be in play during the game. The crowd-anticrowd theory gives a physically transparent quantitative theory of the observed features in this regime, as well as in the inefficient regime. The crowd-anticrowd theory is based on the fact that it is the difference in the numbers of agents playing a strategy R and the corresponding anticorrelated strategy $\overline{R}$ that plays the most important role in the understanding of the fluctuations and hence performance of the whole population.

The crowd-anticrowd theory is a microscopic approach in the sense that it follows the strategy play of the agents in the population. While this leads to microscopically correct equations, in practice these equations are evaluated by simply time averaging over the path taken by the strategy rankings. A naive time averaging over all histories becomes more difficult to implement as the number of ties in strategy scores increases, since such ties affect the number of agents playing a given ranking of strategy and hence must be incorporated explicitly in the time average. The effect of strategy ties becomes increasingly important as m decreases, i.e., size of strategy space decreases, since the chance of ties arising will then increase. In the efficient phase at very low m, therefore, there are frequent ties in the strategy performance [4,21], and hence the time averagings within the crowd-anticrowd theory require special care. In the present paper, we present a complementary theoretical treatment for this regime in order to explicitly account for such strategy ties. The resulting theory amounts to a nontrivial reorganization of the time averagings within the crowd-anticrowd theory. Like the original crowd-anticrowd theory, it is applicable to both non-networked and networked populations [21,22]. The theory we present is based on the idea of following the patterns of the ranking of strategies as the game evolves in time, without knowing the details of the ranking of each strategy. The effects of tied strategies are taken into account by considering the number of strategies belonging to each rank as the game evolves. To illustrate the applications of the theory, we show that the theory explains the features observed in numerical results of a networked version of MG. The theory also allows the evaluation of the success rate of agents with a given number of connected neighbors. The latter is important in the study of the functionality of an underlying network [23] in a competing population. Since analytic techniques such as the replica trick fail in the efficient phase of the MG, we focus the applications of our theory specifically on this regime. The present theory thus provides a framework for analytic treatment of nontrivial collective dynamics in agent-based models of competing populations.

The plan of the paper is as follows. In Sec. II, we define the MG in non-networked and networked populations. In Sec. III, we discuss the different ranking patterns based on performance of the strategies as the game evolves, and the fraction of strategies in each rank. The number of agents using a strategy belonging to a particular rank is derived for both non-networked and networked MG in Sec. IV. In Sec. V, we apply the theory to derive an expression for the mean success rate in the efficient phase as a function of the connectivity in the population and compare results with those obtained by numerical simulations. An alternative way to study the mean success rate is to decompose the population into agents with different numbers of connected neighbors. An expression for the mean success rate of agents with k connected neighbors is derived in Sec. VI. Results are in agreement with numerical simulations. Section VII gives a discussion on the limit of validity of the theory and shows how the theory can be extended to study BAR models at high connectivity.

II. THE MINORITY GAME

The basic MG [5] comprises of N agents competing to be in a minority group at each time step. The only information available to the agents is the history. The history is a bit string of length m recording the minority option for the most recent m time steps. There are a total of 2^m possible history bit strings. For example, $m=2$ has $2^2=4$ possible global outcome histories: 00, 01, 10, and 11. At the beginning of the game, each agent picks s strategies, with repetition allowed. They make their decisions based on their strategies. A strategy is a look-up table with 2^m entries giving the predictions for all possible history bit strings. Since each entry can either be 0 or 1, the full strategy pool contains 2^{2^m} strategies. Adaptation is built in by allowing the agents to accumulate a merit (virtual) point for each of her s strategies as the game proceeds, with the initial merit points set to zero for all strategies. Strategies that predicted the winning (losing) action at a given time step, are assigned (deducted) one (virtual) point. At each turn, the agent follows the prediction of her best-scoring strategy. A random choice will be made for tied strategies.

The networked minority game [21,22] explores the *functionality* of an underlying network in the context of a population competing for limited resource. At each time step t, each agent (node) decides on one or two options, as in the basic MG. Each agent decides in light of (i) *global information* which takes the form of the history of the m most recent global outcomes as in the basic MG, and (ii) *local information* obtained via network connections. The connections here need not be physical—it only matters that the connected neighbors are those with whom an agent can communicate. At each time step, each agent compares the score of its own best-scoring strategy (or strategies) with the highest-scoring strategy (or strategies) among the agents to whom he is connected. The agent adopts the action of whichever strategy is highest scoring overall, using a coin toss to break any ties. The network can be a classical random network or take on the geometry of a growing scale-free network [23]. For simplicity, we here assume a random network, where the connection between any two agents (i.e., nodes) exists with a probability p. Numerical results [21] show that the presence of connections lowers the global performance of the population, while ensuring fairness by lowering the spread in the success rates among the agents. Here, we aim at formulating a theory that can be applied to explain the features observed in the numerical simulations. Since the effect of such connections is typically to increase the chances of strategy ties, particularly at low m, this motivates the present theory's approach of tracking strategy patterns in time.

III. RANKING THE STRATEGIES

The two key ingredients for the present theory are (i) the patterns of strategy rankings according to performance, based on the strategies' virtual points as the game proceeds; and (ii) the fraction of strategies in each rank for each ranking pattern. In the following two subsections, we discuss these two points. The discussion is valid for populations either with or without connections.

A. Ranking pattern

As a particular run of a given game evolves, the pattern of strategy rankings also evolves. The instantaneous strategy ranking depends on the number of history bit strings that have occurred an *odd* number of times and the next outcome will depend on whether the current history bit string has occurred an odd or even number of times. Both of these factors are important in the calculation of the mean success rate of the population.

Suppose we are at a given moment in the run of a game. Let μ be the current history bit string that the agents are using for decisions. Let $\{t^{\nu}_{odd}\}$ be the set of turns (i.e., time steps) so far in which a history ν has occurred an odd number of times (including the initial history bit string that starts the game) and $\{t^{\nu}_{even}\}$ be the set of turns so far in which a history ν has occurred an even number of times [24]. For small values of m, i.e., in the efficient phase of the MG, the outcome time series exhibits the feature of antipersistence or double periodicity [10–12,25,26]. This feature implies that all history bit strings occur with equal probabilities. It means that for a current history μ based on which the agents decide, if the winning side is η (η can be 0 or 1) when $t\in\{t^{\mu}_{even}\}$, the outcome is $1-\eta$ with probability unity in the next occurrence of μ. It follows that no strategies could perform better than the others in an average over time, and the virtual points (VPs) of the strategies cannot show a runaway behavior, i.e., the VPs of strategies will not keep on increasing or decreasing. This property is intimately related to the fact that the Eulerian trail is an underlying quasiattractor of the game in this efficient regime [26]. By focusing on whether a history has occurred an odd or even number of times during a run, we are picking out what is essentially the most important aspect of the outcome series.

For a particular turn t, we define the ranking of the strategies according to their performance up to that point in time based on the VPs of the strategies. The rank-1 strategy or strategies have the highest VPs. The rank-2 strategies are the second best-performing (having the second highest VPs), and so on. For small m (efficient phase), the ranking pattern of the strategies depends on *the number of histories* that have occurred an odd number of times. It is illustrative to consider an example for the case of $m=2$ where there are four possible histories (00), (01), (10), and (11). At $t=0$, all strategies are assigned the same VP. There is only one rank, called rank 1, of the strategies, with all the strategies belonging to this rank. This is also the case when the system returns to a situation equivalent to $t=0$ after visiting every possible path from one history to another an equal number of times. At $t=1$, let 00 be the corresponding history (without loss of generality, the random seed history is taken to be 00). The agents decide in a random fashion as the history has not occurred before (or has occurred an even number of times before). The outcome would be 1 (or 0) with probability $1/2$. Let the outcome be 1, for example. The history bit string will become 001. Prior to the current $m=2$ bit string of 01, one history bit string (namely, 00) occurred once. The strategies are now divided into 2 ranks with rank 1 including strategies that predict 1 for history 00; and rank 2 including strategies that predict 0 for history 00. The strategy VP pattern thus consists of two ranks corresponding to assigning +1 VP for those strategies in rank 1 and −1 for those in rank 2.

If the outcome is also 1 at $t=2$, the strategies that predict 1 for the history 01 will have a higher VP. Note that the history bit string is now 0011. Prior to the current bit string of 11, two $m=2$ bit strings 00 and 01 occurred once. The strategies will then be divided into three ranks after this time step with rank 1 including strategies that predict 1 for both histories 00 and 01; rank 2 including strategies that predict 1 for one of the two histories 00 and 01; and rank 3 including strategies that predict 0 for both histories 00 and 01. The strategy VP pattern thus consists of three ranks corresponding to a VP of +2 for those strategies in rank 1, 0 for those in rank 2, and −2 for those in rank 3.

If at some time t, the history 01 happens again, i.e., the history occurred an odd number of times prior to the one under consideration, the outcome will be 0 due to the crowd effect as the outcome was 1 in the last occurrence of the history. The rank-1 strategies will lose and the rank-3 strategies will win. As a result, the ranking of the strategies is then reduced to two ranks with rank 1 including strategies that predict 1 for history 00; and rank 2 including strategies that predict 0 for history 00.

It is important to note that for a given m in the efficient phase, there are only a *finite number of patterns* for the ranking of the strategy performance. In general, we have the following result for the strategy performance ranking pattern.

If a number of κ histories occurred an odd number of times, the strategies will be divided into $\kappa+1$ ranks. The ranking is as follows: rank 1 including strategies that predicted the correct outcome for all κ histories concerned; rank 2 including strategies that predicted the correct outcome for $\kappa-1$ histories concerned; $\cdots$; rank l including strategies that predicted the correct outcomes for $\kappa-l+1$ histories concerned; $\cdots$; rank $\kappa+1$ including strategies that predicted the correct outcome for 0 histories concerned.

For a given value of m, $0\leqslant\kappa\leqslant 2^m$ as there are 2^m possible histories. In the efficient phase, while the numbers of occurrence for every history are the same when averaged over a long time, κ ($\kappa=0,1,2,\ldots,2^m$) histories may occur an odd number of times in each time step as the game evolves. Therefore, the current strategy ranking pattern can be characterized by the parameter κ. For a time step corresponding to $\kappa=0$, i.e., all the histories had occurred an even number of times, there is only one rank and all the strategies lie in the same rank since they have tied VPs (zero VPs). In other words, there is only one (i.e., $C_0^{2^m}=1$) way to achieve a ranking pattern that consists only of rank 1.

Next we deduce the probability $P(\kappa)$ of having κ histories occur an odd number of times, without invoking too many known details of the dynamics. Assuming that each history has probability $1/2$ to appear as one that has occurred an odd number of times, then out of a total of 2^m history bit strings, the probability $P(\kappa)$ of having κ histories occur an odd number of times is

$$P(\kappa)=C_{\kappa}^{2^m}\left(\frac{1}{2}\right)^{2^m}=C_{\kappa}^{2^m}/2^{2^m}. \tag{1}$$

As the game evolves, the system maps out a path in the history space [26]. As the game goes from one history to another, it also makes transitions from one strategy performance ranking pattern to another. Interestingly, the resulting ranking pattern can be seen as a set of highly correlated, time-dependent random walks where each walk reflects the temporal dynamics of a given strategy's VPs. The dynamical evolution of this pattern is also of interest in its own right. Note that there may be frequent ties in the strategies' performances. A merit of the present approach is that we take explicit account of possible ties in performance among the strategies by grouping them into the same rank. This is important in the efficient phase where there are frequent tied VPs among the strategies.

B. Fraction of strategies in each rank

The fraction of strategies in a particular rank for a given value of κ can be calculated readily. It turns out that the ratio of the number of strategies in increasing ranks (recall rank 1 corresponds to highest VP) follows the numbers in the Pascal triangle. When all the histories occurred an even number of times, there is only one rank with a fraction unity of strategies, i.e., all strategies, belonging to the rank. If only one history ($\kappa=1$) occurred an odd number of times, there are two ($=\kappa+1$) ranks with half (fraction $1/2$) of the strategies in rank 1 and the other half (fraction $1/2$) in rank 2. The ratio of the fractions of strategies in the two ranks is $1:1$. If two histories ($\kappa=2$) occurred an odd number of times, there are three ($=\kappa+1$) ranks, with a fraction $1/4$ of the strategies in rank 1, $1/2$ in rank 2, and $1/4$ in rank 3. The ratio of the fractions is $1:2:1$. For three histories occurring an odd number of times, there are four ranks with the ratio of fractions of strategies in the ranks given by $1:3:3:1$, and so on. For κ histories occurring an odd number of times, the fraction of strategies in rank 1 is $C_0^\kappa/2^\kappa$, the fraction of strategies in rank 2 is $C_1^\kappa/2^\kappa$, and so on. In general, the fraction of strategies in rank l is $C_{l-1}^\kappa/2^\kappa$, where the denominator comes from $\Sigma_{i=1}^{\kappa+1}C_{i-1}^\kappa=2^\kappa$. The ratio of the fractions of strategies in different ranks is thus given by $C_0^\kappa:C_1^\kappa:\cdots:C_{l-1}^\kappa:\cdots:C_\kappa^\kappa$, which are the numbers in the Pascal triangle.

IV. NUMBER OF AGENTS USING A BEST STRATEGY BELONGING TO RANK l

A. Nonconnected population

Consider the case of a nonconnected population, i.e., basic MG or $p=0$ in a networked MG. As an agent uses the best-scoring strategy up to the moment of making a decision, he will use the strategy with the lowest rank among the s strategies that he was randomly assigned at the beginning of the game.

Let κ be the number of histories that occurred an odd number of times. It is convenient for later discussion to introduce the probability

$$\alpha_j^\kappa \equiv \frac{1}{2^\kappa}\sum_{l=j+1}^{\kappa+1} C_{l-1}^\kappa = \frac{1}{2^\kappa}\sum_{l=j}^{\kappa} C_l^\kappa, \tag{2}$$

that an agent holds a strategy with performance *worse than* rank j. For an agent using a rank-1 strategy to decide, he must possess at least one rank-1 strategy. This happens with a probability $1-(\alpha_1^\kappa)^s$, where $(\alpha_1^\kappa)^s$ is the probability that the agent holds s strategies that are all worse than rank 1.

Let N_l be the number of agents who hold a strategy in rank l as their best strategy, for a given κ. In the basic MG, this is also the number of agents who will use a strategy in rank l to decide their action. For a population of N agents, it follows that for given κ

$$N_1 = N[1-(\alpha_1^\kappa)^s]. \tag{3}$$

Similarly, for an agent using a rank 2 strategy, he must hold at least one rank 2 strategy *and* must not hold any rank 1 strategy. Therefore,

$$N_2 = N[(\alpha_1^\kappa)^s-(\alpha_2^\kappa)^s]. \tag{4}$$

In general the number of agents holding a rank l strategy ($l=1,2,\ldots,\kappa$) as their best strategy is given by

$$N_l = N[(\alpha_{l-1}^\kappa)^s-(\alpha_l^\kappa)^s], \tag{5}$$

with $\alpha_0^\kappa=1$ as given by Eq. (2). For $l=\kappa+1$,

$$N_{\kappa+1} = N(\alpha_\kappa^\kappa)^s. \tag{6}$$

As an example, take $s=2$, $N=101$, and consider a moment in the game corresponding to $\kappa=4$. Hence we have $\kappa+1=5$ ranks. The ratio of strategies in these ranks is $1:4:6:4:1$. The average number of agents using strategies in each rank in these turns is given by $N_1=12.23$, $N_2=41.03$, $N_3=37.88$, $N_4=9.47$, and $N_5=0.39$. These numbers change with time as the game evolves to time steps with different values of κ. Knowing the number of agents using each rank of strategies, it is then possible to evaluate analytically the average number of agents making a particular decision and the mean success rate of the agents, as we shall discuss in later sections.

B. Networked population

Let p be the probability that two randomly chosen agents are connected. For $p\neq 0$, the agents may decide based on a strategy that they do not hold. As a result, the number of agents who actually *use* a strategy for decision in a particular rank is, in general, *not* equal to the number of agents N_j who *hold* a best-scoring strategy belonging to that rank [21]. The number of agents $\tilde{N}_l(p)$ who decide by using a rank-l strategy can formally be expressed as a sum of two terms

$$\tilde{N}_l(p) = \overline{N_l} + \sum_{j=l+1}^{\kappa+1}\Delta N_{jl}, \tag{7}$$

where $\overline{N_l}$ is the number of agents who hold a rank-l strategy as their best-performing strategy *and* are not linked to agents with a better (hence lower ranking) performing strategy, and the second term represents all those using a rank-l strategy due to the presence of links. Writing $q=1-p$, $\overline{N_l}$ is then given by

$$\overline{N_l} = N_l q^{\Sigma_{i=1}^{l-1}N_i}, \tag{8}$$

with N_l given by Eqs. (5) and (6). In Eq. (7), ΔN_{jl} is the number of agents who hold a rank-j strategy as their best performing strategy, but they use a rank-l strategy for decision because they are linked to agents carrying such a strategy. Note that $j>l$ because an agent will use the best-performing strategy among his own strategies and his connected neighbors' strategies in our networked MG model. Now consider an agent who does not hold a strategy in rank l but *uses* a rank-l strategy held by one of his neighbors. This happens only when (i) he is not linked to any agent who holds a strategy better than rank l (the probability is thus $q^{\Sigma_{i=1}^{l-1}N_i}$) *and* (ii) he is linked to *at least* one agent who holds a rank-l strategy [the probability is $(1-q^{N_l})$]. Hence we have

$$\Delta N_{jl} = N_j(q^{\Sigma_{i=1}^{l-1}N_i})(1-q^{N_l}). \tag{9}$$

Equation (7) for $\tilde{N}(p)$, coupled with $\overline{N_l}$ given by Eq. (8), ΔN_{jl} given by Eq. (9), and N_j given by Eqs. (5) and (6), gives the

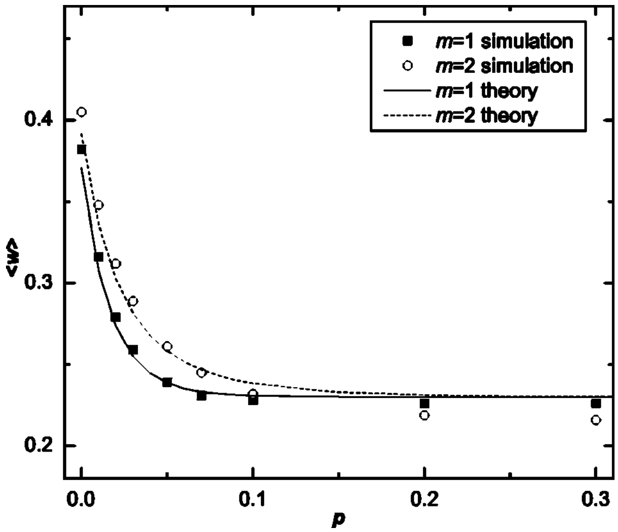

FIG. 1. The mean success rate $\langle w\rangle$ of the agents as a function of connectivity p for $m=2$ and $m=1$. Other parameters are $N=101$ and $s=2$. The symbols are results obtained by numerical simulations and the lines are analytic results obtained by using Eq. (14). Results of numerical simulations represent an average over 1000 different realizations of the system at each value of p.

number of agents who use a strategy in rank l for deciding their action in a connected population.

V. APPLICATION: MEAN SUCCESS RATE

The mean success rate $\langle w\rangle$ (or mean wealth) of the agents is the average number of winners per agent per turn. This quantity reflects the global performance of the population as a whole. This quantity is also closely related to the fluctuations (or standard deviation) in the number of agents choosing a particular option as the game proceeds. A smaller fluctuation implies a higher mean success rate. Figure 1 shows $\langle w\rangle$ as a function of connectivity p obtained by numerical simulations for $m=1$ and $m=2$ (symbols) in a population of $N=101$ agents with $s=2$ strategies per agent. As p increases, $\langle w\rangle$ decreases, together with a drop in the spread of the success rates among the agents [21]. Thus in the networked MG model, while higher connectivity ensures fairness, the efficiency also decreases. Qualitatively, the drop in $\langle w\rangle$ comes about from the enhanced crowd effect as p increases. Here, we derive an expression for the mean success rate as a function of connectivity p. Consider a time step t corresponding to κ histories having occurred an odd number of times. Given this, t may belong to $\{t^{\mu}_{even}\}$ or $\{t^{\mu}_{odd}\}$ for the particular history bit string μ that the population is facing when making a decision, since there are $2^m-\kappa$ histories which have occurred an even number of times.

If $t\in\{t^{\mu}_{odd}\}$, the mean number of agents choosing the last winning option of the corresponding history is given by

$$A_{odd}(\kappa)=\sum_{l=1}^{\kappa+1}\tilde{N}_l(p)\left(\frac{\kappa-l+1}{\kappa}\right). \qquad (10)$$

This is because the rank-l strategies must have made the correct predictions for $\kappa-l+1$ out of the κ histories concerned. Thus, the agents using a rank-l strategy have a probability $(\kappa-l+1)/\kappa$ of choosing the previous winning option for the history μ based on which every agent decides. Due to crowd effect, this is also the probability that the agents using a rank-l strategy lose. Therefore, they will win with a probability $1-(\kappa-l+1)/\kappa=(l-1)/\kappa$. The mean success rate $w_{odd}(\kappa)$ for a given κ and $t\in\{t^{\mu}_{odd}\}$ is

$$w_{odd}(\kappa)=\sum_{l=1}^{\kappa+1}\left(\frac{\tilde{N}_l(p)}{N}\right)\left(\frac{l-1}{\kappa}\right). \qquad (11)$$

If $t\in\{t^{\mu}_{even}\}$, the agents decide randomly and the mean number of agents choosing a particular option is $N/2$. In this case, the probability of having n agents choose a particular option is

$$P_n=C^N_n/2^N, \qquad (12)$$

as every agent has two options. For the MG, the winners are those in the minority group. There are n winners for $n<(N-1)/2$ and $(N-n)$ winners for $n\geqslant(N+1)/2$. The mean success rate w_{even} for $t\in\{t^{\mu}_{even}\}$ is then given by

$$w_{even}=\sum_{n=0}^{(N-1)/2}P_n\frac{n}{N}+\sum_{n=(N+1)/2}^{N}P_n\frac{N-n}{N}. \qquad (13)$$

We note that one may also make the crude approximation that $w_{even}=1/2$, without taking into account the fluctuations in the number of agents making identical decisions.

Given a value of κ, i.e., there are κ histories which have occurred an odd number of times and $2^m-\kappa$ histories which occurred an even number of times, the probability of having a time step $t\in\{t^{\mu}_{odd}\}$ is $\kappa/2^m$. The probability of having a time step $t\in\{t^{\mu}_{even}\}$ is $(1-\kappa/2^m)$. The mean success rate $\langle w\rangle$ is obtained by averaging over the probabilities of having $t\in\{t_{odd}\}$ and $t\in\{t_{even}\}$ for given κ and then averaging over the probability of having κ odd-occurring strategies. The mean success rate is then formally given by

$$\langle w\rangle=\sum_{\kappa=0}^{2^m}P(\kappa)\left(\frac{\kappa}{2^m}w_{odd}(\kappa)+\left(1-\frac{\kappa}{2^m}\right)w_{even}\right), \qquad (14)$$

with $P(\kappa)$ given by Eq. (1). Equation (14) is a general expression for the mean success rate. It is valid for both non-networked and networked populations. Figure 1 compares the analytic results (lines) for $\langle w\rangle$ from Eq. (14) as a function of p for different values of $m=1$ and $m=2$. The results are in very good agreement with results obtained by numerical simulations. The present formalism also provides a physically transparent picture for the drop in $\langle w\rangle$ with p. Since there is no single strategy that consistently outperforms the others, those instantaneously better performing strategy or strategies have a higher chance of losing in immediate time steps. Therefore, forcing the agents to follow the better-performing strategy of their connected neighbors actually *lowers* their mean success rate. The deviation of the analytic results from the numerical results at high connectivity p comes about from the enhancement in the so-called market-impact effect. In the MG, the action of an agent in a time step lowers her chance of winning in that time step. While this market-impact effect is taken into account satisfactorily by Eq. (13) at low connectivity p, the enhanced connections at high p drive too many agents to follow the strategy of a particular agent and hence the market-impact effect becomes enhanced and cannot be simply accounted for by Eq. (13). We emphasize that the present formalism is closely related to the crowd-anticrowd theory [13,14] in that the agents using a strategy and those using the corresponding anticorrelated partner have different success rates given by the term in the last parentheses in Eq. (11) since the pair of strategies must belong to different rankings.

VI. MEAN SUCCESS RATE OF AGENTS WITH DEGREE k

A useful way to describe the topological properties of a network is the degree distribution, which is the distribution of the number of connected neighbors among the nodes in a network [23,27]. Statistical analysis has revealed that real world networks exhibit degree distributions of various kinds [23,27–29]. For classical random graphs discussed in previous sections, the degree distribution is a Poisson distribution [30]; while for growing networks with preferential attachment in its growth mechanism, the degree distribution exhibits power law behavior [23,27]. While the analysis in the last section suffices for evaluating $\langle w\rangle$ in a random network, it will be useful to develop our formalism by focusing on agents with a given number of neighbors, i.e., a given degree. Here, we aim at studying the mean success rate of agents with degree k in a networked MG.

Consider a particular agent having k links to other agents. Recall that [see Eq. (5)] the probability that an agent holds a rank-l strategy as his best-performing strategy is given by $(\alpha_{l-1}^{\kappa})^s-(\alpha_l^{\kappa})^s$. Note that this is also the probability that his neighbor holds a rank-l strategy as his best performing strategy. Combining these probabilities for an agent and his k neighbors, the probability $\gamma(\kappa,k,l)$ of an agent with k neighbors using a rank-l strategy is

$$\gamma(\kappa,k,l)=(\alpha_{l-1}^{\kappa})^{(k+1)s}-(\alpha_l^{\kappa})^{(k+1)s}. \tag{15}$$

This follows from the fact that an agent who has k links is equivalent to an agent who effectively has $(k+1)s$ strategies in hand, with repetition allowed. Recall that the success rate or winning probability of a rank-l strategy is $(l-1)/\kappa$ for $t\in\{t_{odd}^{\mu}\}$. The success rate of an agent with k links for time steps $t\in\{t_{odd}^{\mu}\}$ is given by

$$w_{odd}(k,\kappa)=\sum_{l=1}^{\kappa+1}\gamma(\kappa,k,l)\frac{l-1}{\kappa}. \tag{16}$$

We should also take into account cases corresponding to $t\in\{t_{even}^{\mu}\}$ for which the mean success rate of an agent is given by w_{even} in Eq. (13). As a result, the success rate of an agent with degree k is given by

$$\langle w(k)\rangle=\sum_{\kappa=0}^{2^m}P(\kappa)\left(\frac{\kappa}{2^m}w_{odd}(k,\kappa)+\left(1-\frac{\kappa}{2^m}\right)w_{even}\right). \tag{17}$$

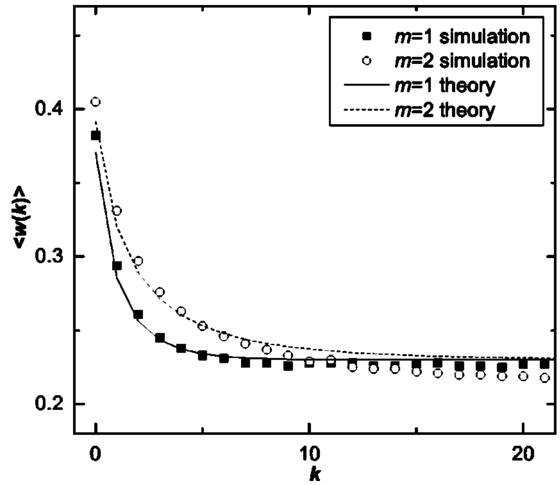

FIG. 2. The mean success rate $\langle w(k)\rangle$ of agents of degree k as a function of k for $m=2$ and $m=1$. Other parameters are $N=101$ and $s=2$. The symbols represent numerical results obtained by carrying out simulations within the range $0\leqslant p\leqslant 0.5$. The lines give the analytic results obtained using Eq. (17).

For the particular case of classical random graphs, the probability of having k links in a system with N nodes (agents) for a given value of connectivity p is given by

$$Y(k)=C_k^{N-1}p^k(1-p)^{N-1-k}. \tag{18}$$

Combining with Eq. (17), the mean success rate in the population with connectivity p is formally given by

$$\langle w\rangle=\sum_{k=0}^{N-1}Y(k)\langle w(k)\rangle. \tag{19}$$

Figure 2 shows the numerical and analytic results of $\langle w(k)\rangle$ as a function of k for $m=1$ and $m=2$. The analytic results are, again, in good agreement with the numerical results. The numerical results are obtained from data in many runs with different values of p ranging from $0\leqslant p\leqslant 0.5$. For a given p, data are obtained for values of k around the mean degree $\langle k(p)\rangle$. We note that, for given degree k and fixed m, $\langle w(k)\rangle$ does not depend on p, i.e., the success rate of isolated agents in a population with $p=0.01$ is the same as that for $p=0.02$ (if isolated agents exist). For the present version of networked MG, the isolated agents, i.e., those without any links, have the highest mean success rate. This drop in $\langle w(k)\rangle$ comes about from the fact agents with connected neighbors effectively hold a substantial portion of the strategies and hence they will join the crowd. By being isolated, one can avoid the crowd and hence achieve a higher success rate. We also checked that $\langle w\rangle$ obtained from Eq. (19) is nearly identical to that obtained by Eq. (18), for small values of m.

VII. DISCUSSION AND EXTENSION TO NETWORKED BAR MODEL

We have formulated a theory applicable to agent-based models in which a population is competing to be in the minority group. The population may be networked or non-networked. The theory is based on the tendency that the system restores itself and avoids the existence of strategies that outperform others. This is the case for the efficient phase in the MG. By invoking the idea that the strategy performance ranking patterns change as the game evolves and that only a finite number of patterns exist, it is possible to study the ranking patterns based on the number of history bit strings that occurred an odd number of times. The fraction of strategies in each rank can be found, together with the number of agents using a strategy of rank l in order to decide. For the case of networked populations, care must be taken to evaluate the number of agents using a strategy of rank l through the connections. An expression for the mean success rate as a function of connectivity p and m can be derived. Results are found to be in good agreement with those obtained by extensive numerical simulations of the networked MG. A geometrical property of networks is the degree distribution. We derived an expression for the mean success rate of agents for a given degree k in a networked MG with the underlying network being a classical random graph. The results are found to be, again, in good agreement with numerical results. The present theory has the merit of taking into account possible ties in the strategies' performance.

The validity of the derived results depends on the assumption that the system passes through quasi-Eulerian paths in the history space in the efficient phases of both the non-networked and networked MG. The details of the dynamics are not important, only that we assume the equal probabilities of the occurrence of the possible outcomes. The formalism can also be applied or extended to other situations that exhibit similar features. To illustrate the idea, we consider the interesting situation in a binary-agent-resource game with *high resource level* in a *highly connected* population, i.e., for high values of p. The BAR model in a networked population represents a networked binary version of Arthur's El Farol problem concerning bar attendance [2–4,7]. In the BAR model, the winning option is no longer decided by the minority side. Instead, there is a general global resource level L $(L<N)$ which is not announced to the agents. At each time step t, each agent decides upon two possible options: whether to access resource L (action +1) or not. The two global outcomes at each time step, "resource overused" and "resource not overused," are denoted as 0 and 1. If the number of agents $n_{+1}[t]$ choosing action +1 exceeds L (i.e., resource overused and hence global outcome 0) then the $N-n_{+1}[t]$ abstaining agents win. By contrast, if $n_{+1}[t]\leq L$ (i.e., resource not overused and hence global outcome 1) then these $n_{+1}[t]$ agents win.

Numerical results for a high-resource-level BAR model show interesting features as a function of the connectivity p. Figure 3 shows the dependence on the mean success rate for $L=90$ in an $N=101$ population as a function of p. For $L>3N/4$ in a nonconnected population $(p=0)$, the system is in a frozen state in the sense that the outcome is persistently 1, i.e., the resource is persistently not overused with $3N/4$ winners per turn. It is observed that as p increases, the system moves away from the frozen state [21]. A high-p limit is eventually reached corresponding to a state of antipersistence

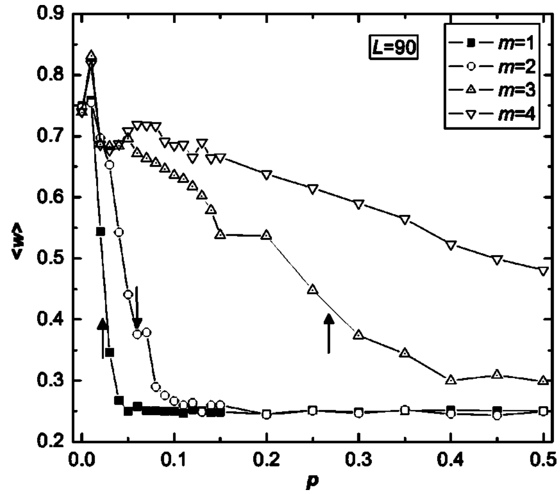

FIG. 3. The mean success rate $\langle w\rangle$ as a function of connectivity p in the BAR model at high resource level $(L=90)$ for $m=1$, 2, 3, 4 obtained by numerical simulations. The lines are guides to the eye. Other parameters are $N=101$ and $s=2$. Results of numerical simulations represent an average over 1000 different realizations of the system at each value of p. The arrows indicate the estimate of $p_c(m)$ using Eq. (20), above which the system goes into a high-p state.

or double periodicity, characterized by an outcome time series with equal probability for the two possible outcomes and a mean success rate slightly lower than 1/4. In particular, it is observed that the value of p [denoted by $p_c(m)$] above which the system reaches the high-p state, depends sensitively on m and increases with m.

The present theory can be extended to estimate $p_c(m)$. To proceed, we propose a criterion that the system is antipersistent only if $A_{odd}(\kappa)>L$ for *all* κ. This can be understood easily since antipersistence implies that for $t\in\{t_{odd}^{\mu}\}$ for the history μ concerned, the outcome will be opposite to that in the last occurrence of the history. However, in the BAR model, for $A_{odd}(\kappa)<L$, the winning option in the last occurrence of the history wins again, and the system ceases to be antipersistent. We further note that $A_{odd}(\kappa)$ is a monotonically decreasing function of κ, with a minimum at $\kappa=2^m$ when all the possible histories occurred an odd number of times. This behavior follows from Eq. (10), and we have also checked it against numerical results.

For a given high resource level L, as p decreases from the high-p state, the difference between $A_{odd}(\kappa)$ and L drops. Eventually when $A_{odd}(\kappa)<L$, the system is no longer antipersistent for some κ. As $A_{odd}(\kappa)$ takes on its minimum value at $\kappa=2^m$, an estimate on the breakdown of antipersistent behavior is then given by the condition

$$A_{odd}(2^m)=L. \tag{20}$$

Equation (20) can be used to estimate the critical value $p_c(m)$ for fixed resource level. To test the validity, we take a system of $N=101$, $L=90$, and $s=2$ (see Fig. 3). The values of p_c turn out to be $p_c=0.0220$ for $m=1$, $p_c=0.0592$ for $m=2$, $p_c=0.2738$ for $m=3$, and $p_c=0.9988$ for $m=4$, as marked by the arrows in Fig. 3. The results capture the trend that $p_c(m)$ increases with m. For $m=5$, our estimate shows that the system cannot achieve an antipersistent high-p state even if $p=1$, a result again consistent with numerical results [21]. Similarly, one may vary the resource level L at given p and Eq. (20) can be used to estimate the critical resource level $L_c(m)$ above which the system starts to deviate from an antipersistent state.

The formalism can also be applied to a non-networked BAR game with resource level $L \gtrsim N/2$, for which the outcome series and thus the history series also exhibit antipersistence or doubly periodic features [8]. While we have illustrated the validity of the present theory by focusing on models for which the history bit strings are visited with equal probability as the system evolves, we stress that the idea of analyzing agent-based models through the ranking of the performance of the strategies or groups of strategies and the number of agents using a strategy of a certain rank is general. The formalism can be readily extended to treat cases in which the outcome (hence history) time series shows known features other than antipersistence. Depending on the model under consideration, the theory can be suitably modified by making use of the statistics in the outcomes series of the model to work out the probability of occurrence of each history bit string and the ranking in the strategies' performance. For example, the theory can be modified to study each of the many states that a high-resource-level BAR game passes through from the frozen state at $p=0$ to the high-p state as the connectivity varies. The starting point is to give a known outcome series, e.g., 11101110… for $m=1$ and 11111101111110… for $m=2$ just off the frozen state. In these cases, only a portion of the whole history space is being explored by the system and the system does not show antipersistent features. Even so, once the pattern of history time series is known, the part of the full history space that matters is also known and thus the ranking pattern of the strategies can be worked out [31]. Similar situations also happen in the networked BAR model [21] with high resources [32]. There are other models, e.g., the majority game, for which the equations derived here specifically for the MG cannot be applied directly. Typically in these models some strategies have runaway VPs and hence the assumption of antipersistence in driving the results in the present work breaks down. We stress that, even for these models, a proper starting point in analyzing the results is, as in the present approach, to work out the strategy performance ranking pattern. It is worth pointing out that "unexpected changes" [33] such as crashes and bubbles in markets can be modeled within the framework of the minority game by including complications such as a finite time horizon in evaluating the strategy performance [2,34] and a confidence level for the agents to evaluate their cumulative performance in deciding whether to participate in each turn of the game [2,35,36]. These additional considerations lead to more complicated probabilities of occurrence of history bit strings and hence more involved strategy ranking patterns. While it will be hard to calculate the results of these models directly, the ideas in the present approach can be extended to analyze numerical results systematically by working out the strategy ranking pattern from the outcome series. For other models in which imitation through networking does not result in a simple enlargement in the number of strategies that an agent holds, Eqs. (7)–(9) need to be properly modified.

We have assumed uniform initial conditions in our derivations. There are several possible sources of nonuniformity in the initial conditions that may affect our results. For example, there may be nonuniform distribution of strategies among the agents or intrinsic nonuniformity in the possible strategies being allocated to the agents. In these cases, the strategy ranking pattern can be found by analyzing the outcome series. Quite generally, there will be fewer agents using the better performing strategy or strategies, and hence Eqs. (7)–(9) needed to be modified. Another source of nonuniform initial conditions is a bias in the initial VPs. In this case, it has been shown [26] that the system goes through a transient behavior and relaxes to the steady state VP pattern, with possible persistent (i.e., majoritylike) behavior in the transient. To analyze the transient behavior, it is again important to follow the strategy performance ranking patterns. Using the ideas that the ranking pattern of the strategies and the number of agents using a strategy of a certain rank for decisions play the crucial role in analyzing a wide class of agent-based models, the present formalism can also be extended to study different variations on the basic MG, such as the thermal MG [37,38] and the MG with biased strategy pools [39]; and to different versions of the networked MG in which neighboring agents compare their wealth instead of strategy performance [40,41].

ACKNOWLEDGMENTS

This work was supported in part by the Research Grants Council of the Hong Kong SAR Government under Grant No. CUHK 4241/01P. The authors would like to thank S. Gourley, S. C. Choe, and K. P. Chan for useful discussions.

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